Exercise 38

Given vectors $\mathbf{a}$ and $\mathbf{b}$, do the equations $\mathbf{x} \times \mathbf{a} = \mathbf{b}$ and $\mathbf{x} \cdot \mathbf{a} = \|\mathbf{a}\|$ determine a unique vector $\mathbf{x}$? Argue both geometrically and analytically.

Solution

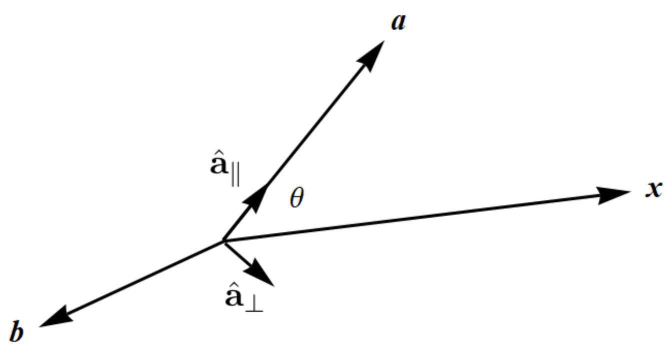
Take the magnitude of both sides of the equation involving the cross product.

$$\begin{aligned} \mathbf{x} \cdot \mathbf{a} &= \|\mathbf{a}\| \\ \|\mathbf{x} \times \mathbf{a}\| &= \|\mathbf{b}\| \end{aligned}$$

Suppose that θ is the angle between $\mathbf{x}$ and $\mathbf{a}$.

$$\begin{aligned} \|\mathbf{x}\|\|\mathbf{a}\| \cos\theta &= \|\mathbf{a}\| \\ \|\mathbf{x}\|\|\mathbf{a}\| \sin\theta &= \|\mathbf{b}\| \end{aligned} \quad \rightarrow \quad \begin{aligned} \|\mathbf{x}\| \cos\theta &= 1 \\ \|\mathbf{x}\| \sin\theta &= \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \end{aligned}$$

From the equation $\mathbf{x} \times \mathbf{a} = \mathbf{b}$, the vector $\mathbf{b}$ is perpendicular to the plane containing $\mathbf{x}$ and $\mathbf{a}$.



A basis for this plane can be written in terms of $\hat{\mathbf{a}}_{\|}$, a unit vector parallel to $\mathbf{a}$, and $\hat{\mathbf{a}}_{\perp}$, a unit vector perpendicular to $\mathbf{a}$ that also lies in the plane.

$$\hat{\mathbf{a}}_{\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \qquad \hat{\mathbf{a}}_{\perp} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$$

Write $\mathbf{x}$ in terms of these basis vectors.

$$\mathbf{x} = x_1 \hat{\mathbf{a}}_{\|} + x_2 \hat{\mathbf{a}}_{\perp}$$

To determine x_1, take the dot product of both sides with $\hat{\mathbf{a}}_{\|}$.

$$\begin{aligned} \mathbf{x} \cdot \hat{\mathbf{a}}_{\|} &= x_1(\hat{\mathbf{a}}_{\|} \cdot \hat{\mathbf{a}}_{\|}) + x_2(\hat{\mathbf{a}}_{\perp} \cdot \hat{\mathbf{a}}_{\|}) \\ \|\mathbf{x}\| \cos\theta &= x_1(1) + x_2(0) \\ 1 &= x_1 \end{aligned}$$

To determine x_2, take the dot product of both sides with $\hat{\mathbf{a}}_{\perp}$.

$$\begin{aligned} \mathbf{x} \cdot \hat{\mathbf{a}}_{\perp} &= x_1(\hat{\mathbf{a}}_{\|} \cdot \hat{\mathbf{a}}_{\perp}) + x_2(\hat{\mathbf{a}}_{\perp} \cdot \hat{\mathbf{a}}_{\perp}) \\ \|\mathbf{x}\| \sin\theta &= x_1(0) + x_2(1) \\ \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} &= x_2 \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{x} &= x_1\hat{\mathbf{a}}_{\parallel} + x_2\hat{\mathbf{a}}_{\perp} \\
&= 1\frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|} \\
&= \frac{\mathbf{a}}{\|\mathbf{a}\|} + \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\frac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|\sin 90^\circ} \\
&= \frac{\|\mathbf{a}\|\mathbf{a} + \mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\|^2}.
\end{aligned}$$